ARTICLE

Prime Labeling of Some Graphs with Eisenstein Integers

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ABSTRACT

A graph on n vertices is said to admit a prime labeling if the vertices can be labeled with the first n natural numbers in such a way that two adjacent vertices have relatively prime labels. In this paper, we define an order on the set of Eisenstein integers to extend the notion of prime labeling of graphs to the set of Eisenstein integers. Properties of the ordering are studied to come up with prime labelings of some families of graphs such as the flower, wheel, centipede, and double broom graphs.

INTRODUCTION

Graph labeling, an assignment of integers to the vertices (or edges) of a graph, is used in coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, and database management [1]. Some daily life applications of graph labeling include scheduling schemes for guard posts and finding the shortest route between two places as exhibited by GPS systems, among others. Many techniques of labeling a graph have been examined over the last decades. For a comprehensive account on the history and results of graph labeling, the reader is referred to [2].

Prime labeling, a particular method of graph labeling, was introduced by Entringer in the 1980s. If V is the set of vertices of a graph G and $|V| = n$, then G has a prime labeling if the vertices can be labeled with the first n integers such that the vertices u and v have relatively prime labels whenever uv is an edge of G. Entringer conjectured that all trees admit a prime labeling but this has only been proven for small trees [3, 4, 5]. Nonetheless, many classes of graphs are known to admit prime labeling such as paths, cycles, stars, double stars, caterpillars, complete binary trees, binomial trees, spiders, palm trees, olive trees, banana trees, etc.

Klee et al. (2016) showed that several families of trees admit a prime labeling using the Gaussian integers [6]. An order on $\mathbb{Z}[i]$

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called the spiral ordering was introduced which made the labeling using the first n Gaussian integers possible. Shrimali and Singh (2021), as a continuation of the work by Klee et al., explored more types of graphs and proved that they admit prime labelings using the Gaussian integers [7]. In this present work, we define an order on the set of Eisenstein integers and use properties of this ordering to obtain a prime labeling for some families of graphs using the Eisenstein integers.

Preliminaries

Eisenstein integers are complex numbers of the form $a + b\omega$ where $\omega = e^{\frac{i2\pi}{3}}$ and $a, b \in \mathbb{Z}$. If $b = 0$, the Eisenstein integer $a + b\omega$ is said to be *rational*. The set of all Eisenstein integers is denoted by $\mathbb{Z}[\omega]$, and like $\mathbb{Z}$ and $\mathbb{Z}[i]$, is a unique factorization domain. The units of $\mathbb{Z}[\omega]$ are $\pm 1, \pm\omega$, and $\pm(\omega + 1)$. An *associate* of an Eisenstein integer α is αu where u is any unit in $\mathbb{Z}[\omega]$. The *norm* of $a + b\omega$, denoted $N(a + b\omega)$, is given by $a^2 - ab + b^2$. This norm function N on $\mathbb{Z}[\omega]$ is multiplicative, i.e., $N(\alpha\beta) = N(\alpha)N(\beta)$, for any $\alpha, \beta \in \mathbb{Z}[\omega]$. We say that an Eisenstein integer is *even* if the norm is even, and *odd* otherwise. Note that we use the term *parity* in this paper to denote the property of being odd or even of an Eisenstein integer.

Now, an Eisenstein integer is *prime* if and only if its only nonunit divisors are its associates. In [8], it was shown that a prime Eisenstein integer falls under one of the following types: (i) it is an associate of $-1 + \omega$; (ii) it is a rational prime $p \equiv 2 \pmod 3$; or (iii) it is an Eisenstein integer $a + b\omega$ where $N(a + b\omega) = p \equiv 1 \pmod 3$. Moreover, two Eisenstein integers α and β are *relatively prime* or *coprime* if and only if their common divisors are the units in $\mathbb{Z}[\omega]$.

To illustrate, 5 and $3 + 2\omega$ are prime Eisenstein integers. However, $7 + 7\omega$ is not prime in $\mathbb{Z}[\omega]$ because $7 + 7\omega = (3 + \omega)(3 + 2\omega)$. Since 7 and $7 + 7\omega = 7(1 + \omega)$ are associates, then 7 is also not a prime Eisenstein integer. Moreover, 2 is relatively prime to $3 + 2\omega$ but 7 and $3 + 2\omega$ are not relatively prime since they have a nonunit common divisor, namely $3 + 2\omega$.

A *graph* $G = (V, E)$ is composed of two finite sets V of vertices and E of unordered pairs of distinct vertices called edges. The order of a graph G is given by $|V|$. We write $uv \in E$ to mean that there is an edge in E that joins the vertices $u, v \in V$. The *degree* of a vertex v is n, denoted $d(v) = n$, if there are n edges incident to v. Now, a *path* is a sequence of edges joining a sequence of distinct vertices. We say that a graph is *connected* if it does not contain two vertices which are not connected by a path, and *unconnected* otherwise. On the other hand, a *cycle* of a graph is a sequence of at least three distinct edges which begins and ends at the same vertex. We say that a graph is *cyclic* if it contains at least one cycle and *acyclic* otherwise. A *tree* is defined as a connected acyclic graph while a *forest* is a graph that is unconnected and acyclic. We call a vertex v an *internal node* of a tree if $d(v) > 1$ while we call v a *leaf* or *endvertex* of a tree if $d(v) = 1$.

MATERIALS AND METHODS

Ordering of the Eisenstein Integers

Similar to the Gaussian integer case, Eisenstein integers possess no natural ordering. Hence, we need to introduce an ordering to define what we mean by "the first n Eisenstein integers" before we can proceed with the prime labeling of graphs. Since associates of a given Eisenstein integer occur every 60° rotation about the origin, we will only consider the Eisenstein integers that lie in the sector $\left[0, \frac{\pi}{3}\right)$. This choice is purely out of convenience as it is easier to work with Eisenstein integers with positive components. This is comparable to the choice of Gaussian integers lying in Quadrant I as done by Klee et al. in [6].

Definition 3.1. (Diagonal Ordering of Eisenstein Integers) The nth Eisenstein integer γ_n is defined recursively as follows: $\gamma_1 = 1$, and for $n \in \mathbb{N}$, if $\gamma_n = a + b\omega$, we have

$$\gamma_{n+1} = \begin{cases} \gamma_n + 1, & \text{if } a \equiv 1 \pmod 2 \text{ and } b = 0 \\ \gamma_n + \omega + 1, & \text{if } a \equiv 0 \pmod 2 \text{ and } b = a - 1 \\ \gamma_n + \omega, & \text{if } a \equiv 0 \pmod 2 \text{ and } b \neq a - 1 \\ \gamma_n - \omega, & \text{if } a \equiv 1 \pmod 2 \text{ and } b \neq 0. \end{cases}$$

The diagonal ordering of the Eisenstein integers is shown in Figure 1. From this point onward, we take $[\gamma_n]$ to denote the set of the first n Eisenstein integers. Thus, under this ordering, $[\gamma_{10}] = \{1, 2, 2 + \omega, 3 + 2\omega, 3 + \omega, 3, 4, 4 + \omega, 4 + 2\omega, 4 + 3\omega\}$. If γ is the nth Eisenstein integer under the diagonal ordering, we say that γ has index n and write $I(\gamma) = n$.

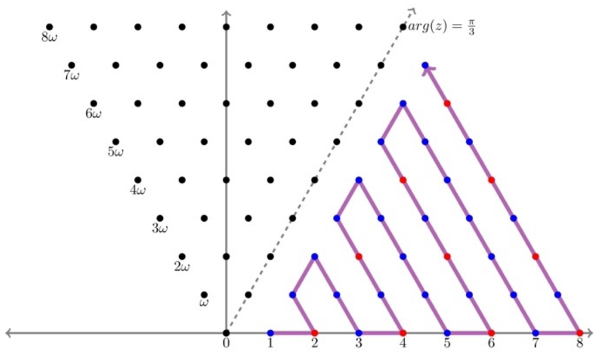

Figure 1: Diagonal Ordering of the Eisenstein Integers.

At a glance, we can immediately deduce that not all properties of the usual ordering of $\mathbb{N}$ are preserved. The norm $N(a + b\omega) = a^2 - ab + b^2$ is odd if and only if at least one of a or b is odd. Thus, consecutive Eisenstein integers under the diagonal ordering may not always alternate parity. For instance, $3 + \omega$ and 3 occur consecutively but both their norms are odd. Nonetheless, some properties of the usual ordering of $\mathbb{N}$ are still preserved, as can be seen in the next section.

This definition of diagonal ordering leads to the following definition of prime labeling using Eisenstein integers.

Definition 3.2. We say that a graph $G = (V, E)$ on n vertices admits an Eisenstein prime labeling if we can label its vertices with the first n Eisenstein integers in the diagonal ordering in such a way that $u, v \in V$ have relatively prime labels whenever $uv \in E$.

Properties of the Diagonal Ordering

Here, we discuss properties of the diagonal ordering. We begin by setting terminologies for portions of the diagonal ordering. *Corners* are the Eisenstein integers occurring at the turning points of the ordering from east to northwest, from northwest to northeast, from northeast to southeast, or from southeast to east. If the corners occur at the real axis, then we call them *real corners*. A *diagonal* (of the ordering) consists of the Eisenstein integers traversed when going northwest or going southeast.

In light of these new terminologies, we note that each diagonal contains exactly one real corner. Hence, we may define the *kth diagonal line* as the diagonal containing the real corner k. Furthermore, *even diagonals* are the diagonals containing even real corners while *odd diagonals* are the diagonals containing odd real corners. Lastly, we define the *initial point* of a diagonal to be the corner when turning northwest or southeast and the *terminal point* of a diagonal to be the corner when turning east or northeast. (The Eisenstein integer 1 is the lone vertex in its diagonal and hence neither an initial nor a terminal point.)

Let $k \in \mathbb{N}$. Then, the kth diagonal line consists of the Eisenstein integers $k, k+\omega, k+2\omega, \dots,\ k+(k-1)\omega$. If k is even, then k occurs as the initial point of the kth diagonal. Otherwise, it occurs as the terminal point. Since the number of Eisenstein integers in all the diagonal lines is given by the sequence $\{1,2,3,\dots\}$, then the index of the terminal point of the kth diagonal line is given by $\frac{k(k+1)}{2}$.

In **Error! Reference source not found.**, odd Eisenstein integers are colored blue while even Eisenstein integers are colored red. Even integers occur only in even diagonals since $N(a+b\omega)$ is an even rational number if and only if both a and b are even. Hence, just like in $\mathbb{Z}$, an Eisenstein integer is even if and only if it is a multiple of 2. Observe also that consecutive Eisenstein integers occurring along even diagonal lines alternate in parity. With this, consecutive even Eisenstein integers occurring along the same diagonal are two indices away from each other. On the other hand, successive even Eisenstein integers occurring along different diagonals are of the form $k+(k-2)\omega$ and $k+2$, where k is an even natural number and are therefore $k+3$ indices away from each other.

In the spiral ordering of Gaussian integers, it is possible for a prime Gaussian integer to be preceded by one of its multiples. However, this cannot happen given the usual order on $\mathbb{N}$ and the diagonal ordering on $\mathbb{Z}[\omega]$.

Lemma 3.1. *Eisenstein primes are not preceded by their multiples in the diagonal ordering.*

Proof. Let γ_q be a prime Eisenstein integer lying on the kth diagonal line. If γ_q is a rational prime p, then $N(\gamma_q)=p^2$. Otherwise, $N(\gamma_q)=p$ where p is a rational prime. If $\gamma_q \mid \alpha$ for some Eisenstein integer α and α is not an associate of γ_q, it follows that $N(\alpha)=mp$ or $N(\alpha)=mp^2$ for some positive integer $m>1$. The smallest possible values for the norm of α for each case are $N(\alpha)=3p$ or $N(\alpha)=3p^2$, respectively. Thus, we only need to show that α lies beyond the diagonal containing γ_q.

Suppose γ_q occurs on the kth diagonal line. Consider the equilateral triangle bounded by the real axis, the line $\arg(z)=\frac{\pi}{3}$ and the kth diagonal line. A perpendicular bisector of this triangle is the line $\arg(z)=\frac{\pi}{6}$. Recall that the set of Eisenstein integers occurring on the kth diagonal is given by $A=\{k, k+\omega, \dots, k+(k-1)\omega\}$. The square roots of the norms of these Eisenstein integers are exactly their distance from the origin. Hence, using the symmetry along the $\frac{\pi}{6}$- axis, we may deduce that for any Eisenstein integer $\beta \in A$, we have $\frac{\sqrt{3}}{2}k \le \sqrt{N(\beta)} \le k$. Since $\gamma_q \in A$, it is enough to show that $\sqrt{N(\alpha)} > k$. But $\frac{\sqrt{3}}{2}k \le \sqrt{N(\gamma_q)}$ implies that $\frac{\sqrt{3}}{2}k \le \sqrt{p} < p$ and so $\frac{3}{2}k \le \sqrt{3p} < \sqrt{3p^2}$. Hence, $k < \frac{3}{2}k \le \sqrt{N(\alpha)}$ and the result follows. ■

Lemma 3.2. *Let α be an Eisenstein integer and u be a unit of $\mathbb{Z}[\omega]$. Then α and $\alpha+u$ are relatively prime.*
Proof. Suppose there exists an Eisenstein integer β satisfying $\beta \mid \alpha$ and $\beta \mid (\alpha+u)$. Then $\beta \mid u=(\alpha+u)-\alpha$ making β a unit. Thus, α and $\alpha+u$ are relatively prime. ■

Corollary 3.3. *Consecutive Eisenstein integers in the diagonal ordering are relatively prime.*

Corollary 3.4. If γ_n, γ_{n+2} are odd Eisenstein integers, then they are relatively prime.

Proof. The difference between odd Eisenstein integers that are two indices away from each other is one of the following: $\pm 1, \pm(1+\omega), \pm 2\omega$. If there exists $\beta \in \mathbb{Z}[\omega]$ dividing both integers, then it must divide their difference. If the difference is a unit, then by Lemma 3.2. , they are relatively prime. Suppose their difference is $\pm 2\omega$. Then $N(\beta)|4$. Hence, $N(\beta) \in \{1,2,4\}$ but $N(\beta)$ is odd by assumption and so $N(\beta)=1$. Hence, β is a unit and the result follows. ■

By Corollaries Corollary **3.3.** and Corollary **3.4.** , we have that consecutive odd Eisenstein integers in the diagonal ordering are relatively prime.

Given an Eisenstein integer $a+b\omega$ with $0 \le b \le a-1$, we find its index under the diagonal ordering. We know $a+b\omega$ occurs on the ath diagonal line. Now, the index of the terminal point of the $(a-1)$th diagonal line is given by $\frac{(a-1)a}{2}$. Hence, we only need to find how many indices away from the initial point of the ath diagonal line $a+b\omega$ is and add it to $\frac{(a-1)a}{2}$. We consider two cases.

(1) If a is even, then the initial and terminal point of the ath diagonal line is the real corner a and $a+(a-1)\omega$, respectively. An Eisenstein integer that follows any Eisenstein integer between a and $a+(a-1)\omega$ in the diagonal ordering is obtained by simply adding ω to the preceding Eisenstein integer. Hence, in this case, b is the number of Eisenstein integers occurring before $a+b\omega$ in the ath diagonal line. This implies that $a+b\omega$ is $b+1$ steps away from the terminal point of the $(a-1)$th diagonal line.

(2) If a is odd, then the initial and terminal points of the ath diagonal line are $a+(a-1)\omega$ and a, respectively. The Eisenstein integers on this diagonal are now obtained by subtracting ω repeatedly from $a+(a-1)\omega$. Thus, there will be $(a-1)-b$ Eisenstein integers preceding $a+b\omega$ on the ath diagonal line in this case.

These results are summarized in the following lemma.

Lemma 3.5. *Let $a+b\omega$ be an Eisenstein integer with $0 \le b \le a-1$. Then the index of $a+b\omega$ under the diagonal ordering is given by*

$$I(a+b\omega)=\begin{cases} \dfrac{(a-1)a}{2}+b+1, & \text{if } a \equiv 0 \pmod 2 \\ \dfrac{(a-1)a}{2}+(a-b), & \text{if } a \equiv 1 \pmod 2. \end{cases}$$

Going in the opposite direction is a little more complicated. If we wish to find the Eisenstein integer γ_n for a fixed index n, we have to first identify the diagonal line where γ_n lies. If we assume that γ_n is found on the kth diagonal line, one can easily verify that $k=\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil$ using the fact that the index of the terminal point of each diagonal is a triangular number. Then, reversing the results of Lemma 3.5. yields the following equations.

Lemma 3.6. *Let n be a fixed positive integer and let $k = \left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil$. Then*

$$\gamma_n = \begin{cases} k + \left(n - 1 - \frac{(k-1)k}{2}\right)\omega, & \text{if } k \equiv 0 \pmod 2 \\ k + \left(k - \left(n - \frac{(k-1)k}{2}\right)\right)\omega, & \text{if } k \equiv 1 \pmod 2). \end{cases}$$

Given an Eisenstein integer $a + b\omega$ where $0 \leq b \leq a - 1$, we can now find the index upon which it occurs in the diagonal ordering, and vice versa. For graph labeling purposes in the next section, it is also useful if we can count the number of even Eisenstein integers that are inside $[\gamma_n]$ for a fixed n.

We already know that even Eisenstein integers occur only along even diagonal lines and that consecutive Eisenstein integers along even diagonal lines alternate parity. It follows that there are exactly $\frac{m}{2}$ even Eisenstein integers along an even diagonal with real corner m. This implies that the number of even Eisenstein integers in successive even diagonals is given by the sequence $\{1,2,3,\ldots\}$. Hence, whenever m is even, the number of even Eisenstein integers contained in $\left[\gamma_{\frac{m(m+1)}{2}}\right]$, where $\gamma_{\frac{m(m+1)}{2}}$ is the terminal point of the mth diagonal line, is given by

$$\sum_{i=1}^{\frac{m}{2}} i = 1 + 2 + 3 + \cdots + \frac{m}{2} = \frac{\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right)}{2} = \frac{m(m+2)}{8}. \qquad (1)$$

Suppose γ_n is in the kth diagonal line and assume k is odd. Then the number of even Eisenstein integers occurring before the initial point up to the terminal point of the kth diagonal line is given by the number of even Eisenstein integers contained in $\left[\gamma_{\frac{(k-1)k}{2}}\right]$, where $\gamma_{\frac{(k-1)k}{2}}$ is the terminal point of the $(k-1)$th diagonal line. By Equation (1), the number of even Eisenstein integers contained in $[\gamma_n]$ is given by

$$\frac{(k-1)(k+1)}{8}. \qquad (2)$$

If k is even, we can use Equation (2) to count the number of even Eisenstein integers occurring before the terminal point of the $(k-1)$th diagonal line. This number is now the same as the number of even Eisenstein integers contained in $\left[\gamma_{\frac{(k-2)(k-1)}{2}}\right]$ which is equal to $\frac{(k-2)k}{8}$. We are now left to count the even Eisenstein integers along the kth diagonal line starting from the initial point up to γ_n.

From Lemma 3.6, we deduced that $\gamma_n = k + \left[n - 1 - \frac{(k-1)k}{2}\right]\omega$ since k is even. This implies that γ_n is exactly $n - \frac{(k-1)k}{2}$ indices away from the initial point of the kth diagonal line. Since consecutive Eisenstein integers alternate parity starting from even along this diagonal, then there are $\left\lfloor \frac{n+1}{2} - \frac{(k-1)k}{4} \right\rfloor$ even Eisenstein integers starting from the initial point up to γ_n. Finally, the number of even Eisenstein integers contained in $[\gamma_n]$ is given by

$$\frac{(k-2)k}{8} + \left\lfloor \frac{n+1}{2} - \frac{(k-1)k}{4} \right\rfloor.$$

These results are formalized in the following theorem.

Theorem 3.7. *Let $n \in \mathbb{N}$ and $k = \left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil$. Then the number $E(\gamma_n)$ of even Eisenstein integers contained in $[\gamma_n]$ is given by*

$$E(\gamma_n) = \begin{cases} \frac{(k-1)(k+1)}{8}, & \text{if } k \text{ is odd} \\ \frac{(k-2)k}{8} + \left\lfloor \frac{n+1}{2} - \frac{(k-1)k}{4} \right\rfloor, & \text{if } k \text{ is even.} \end{cases}$$

Prime Labeling Results

In this section, we consider some families of graphs and show that they admit an Eisenstein prime labeling under the diagonal ordering. We also deal with other families of graphs and give values of n for which they admit an Eisenstein prime labeling using the first n Eisenstein integers. Path, cycle, and star graphs, defined formally below, are easily shown to admit Eisenstein prime labeling by Corollary 3.3. For a vertex v in a graph G, we shall denote by $\ell(v)$ the label of the vertex v.

Definition 4.1. Let $n \in \mathbb{N}$. A *path graph* $P_n = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$.

Definition 4.2. Let $n \geq 3$. A *cycle graph* $C_n = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$.

Definition 4.3. Let $n \geq 3$. A *star graph* $S_n = (V, E)$ is a graph with $V = \{v_1, v_2, \ldots, v_{n+1}\}$ and $E = \{v_1 v_i : 2 \leq i \leq n+1\}$. We call v_1 the *center* of S_n.

To achieve Eisenstein prime labeling for the aforementioned graphs, we simply let $\ell(v_i) = \gamma_i$. By Corollary 3.3. the path graph P_n admits an Eisenstein prime labeling for any n. Additionally, since $\gamma_1 = 1$ is relatively prime to any Eisenstein integer, then the cycle graph C_n and the star graph S_n both admit Eisenstein prime labelings for any n.

Theorem 4.1. *The path graph P_n, the cycle graph C_n, and the star graph S_n all admit Eisenstein prime labelings.*

In graph theory, we say that a tree is *rooted* if there exists a distinguishable vertex where the other parts of the tree originate. The star graph is an example of a rooted tree with the central vertex v_1 as the root. Note that if we remove v_1, we are left with a collection of n isolated vertices which is essentially a forest consisting of n copies of P_1. Thus, given a rooted tree T with root v_1, taking $\ell(v_1) = 1$ gives us a guarantee that as long as we can give a prime labeling for the forest $T - \{v_1\}$, the graph admits an Eisenstein prime labeling. The spider graph and the generalized friendship graph (although not a tree) benefit from this labeling strategy.

Definition 4.4. A *spider tree* is a tree with one vertex of degree ≥ 3 and all other vertices having degree 1 or 2.

Definition 4.5. Let $n \geq 2$ and $m \geq 3$. The generalized friendship graph $f_{m,n}$ is a graph with $n(m-1)+1$ vertices consisting of n copies of the cycle graph C_m meeting at a common vertex.

As what was done to paths, cycles, and stars, we briefly explain why Eisenstein prime labelings always exist for spiders and friendship graphs.

Given a spider graph, let v_1 be of degree j, where $j \geq 3$. Removing v_1 leaves us with paths $P_{k_1}, P_{k_2}, \dots, P_{k_j}$ with lengths $k_1, k_2, \dots, k_j$ respectively. Label v_1 with 1, then label the vertices of P_{k_1} with the next k_1 Eisenstein integers, the vertices of P_{k_2} with the next k_2 Eisenstein integers, and so on. Since consecutive Eisenstein integers are relatively prime and $\ell(v_1) = 1$ is relatively prime to any Eisenstein integer, then the spider tree admits an Eisenstein prime labelling.

The generalized friendship graph is also known as a flower graph where the cycles C_m are regarded as petals. Let v_1 be the common vertex of the n petals. Removing v_1 leaves us with n copies of $P_{\mathrm{m}-1}$. Let $\ell(v_1) = 1$ and label the vertices of the first $P_{\mathrm{m}-1}$ with the next $m-1$ Eisenstein integers, the vertices of the second $P_{\mathrm{m}-1}$ with the next $m-1$ Eisenstein integers, and so on. Using the same reasoning applied for a spider graph, the generalized friendship graph admits an Eisenstein prime labeling.

Theorem 4.2. *Any spider graph admits an Eisenstein prime labeling. Given positive integers* $n \geq 2$ *and* $m \geq 3$, *the generalized friendship graph* $f_{m,n}$ *admits an Eisenstein prime labeling.*

The labeling strategy for the next graph is not as straightforward as the ones utilized for the graphs considered earlier. We note that the way graphs are defined in this paper enables us to conveniently label a vertex v_i by the Eisenstein integer γ_i to come up with an Eisenstein prime labeling. If this is not feasible, then adjustments with the labeling need to be done. This is illustrated in the next graph.

Definition 4.6. Let $n \geq 4$. A *wheel graph* $W_n = (V, E)$ is a graph with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{v_1 v_i : 2 \leq i \leq n\} \cup \{v_i v_{i+1} : 2 \leq i \leq n-1\} \cup \{v_2 v_n\}$.

Theorem 4.3. *For any positive integer* $n \geq 4$*, the wheel graph* W_n *admits an Eisenstein prime labeling.*

Proof. First, note that if v_1 is removed from W_n, what is left is the cycle C_{n-1}. Temporarily label v_i with γ_i for $i \in \{1, 2, \dots, n\}$. Under this labeling scheme, the cycle might not necessarily have relatively prime labels for v_2 and v_n. However, if γ_n is odd, then it is relatively prime to $\gamma_2 = 2$. In this case, the graph admits an Eisenstein prime labeling as $\ell(v_1) = \gamma_1$ is relatively prime to any Eisenstein integer.

If γ_n is even, we consider two cases.

(a) If $\gamma_n \notin \mathbb{Z}$, consider the set $S = [\gamma_n] \setminus \{\gamma_n - 1\}$. Note that if γ_n lies on the kth diagonal line, then $\gamma_n - 1$ is the Eisenstein integer located on the $(k-1)$ th diagonal line, exactly one step to the left of γ_n. Moreover, we note that the "jump" in set S happens between two odd Eisenstein integers two indices away from each other and are thus relatively prime by Corollary 3.4. . Now, we take $\ell(v_n) = \gamma_n - 1$ and label $\{v_1, v_2, \dots, v_{n-1}\}$ consecutively with the Eisenstein integers contained in S. Since γ_n and $\gamma_n - 1$ differ by a unit, then they are coprime. Thus, we arrive at an Eisenstein prime labeling.

(b) If $\gamma_n \in \mathbb{Z}$, we further deal with two subcases.

(i) If γ_n is not divisible by 3, then we let $\ell(v_i) = \gamma_i$ for $1 \leq i \leq n-2$ and take $\ell(v_n) = \gamma_{n-1}$ which is odd, and $\ell(v_{n-1}) = \gamma_n$. This makes γ_{n-2} and γ_n adjacent. But by the definition of our diagonal ordering, γ_{n-2} and γ_n differ by $-1+\omega$. By assumption, the norm of γ_n is not divisible by $3 = N(-1+\omega)$, and thus γ_{n-2} and γ_n are relatively prime. Hence, prime labeling is achieved.

(ii) If γ_n is divisible by 3, then $\gamma_n = 6m$ for some $m \in \mathbb{N}$. We then "insert"' γ_n between the two odd Eisenstein integers $(6m-1) + 3m\omega$ and $(6m-1) + (3m-1)\omega$. If $\gamma_k = (6m-1) + (3m-1)\omega$, then the labeling technique for W_n is more explicitly described as follows:

$$\ell(v_j) = \begin{cases} \gamma_j, & \text{if } 1 \leq j \leq k-1 \\ \gamma_n, & \text{if } j = k \\ \gamma_{j-1}, & \text{if } k < j \leq n. \end{cases}$$

From this labeling, we are now assured that $\ell(v_2) = 2$ and $\ell(v_n) = 6m-1$ are relatively prime.

Now, since $\gamma_{k-1} = (6m-1) + 3m\omega$ and $\gamma_k = (6m-1) + (3m-1)\omega$ are both odd, then any divisor of γ_n common with either γ_{k-1} or γ_k must divide $3m$. If r is an Eisenstein integer such that $r \mid 3m$ and $r \mid \gamma_{k-1}$, then r divides any linear combination of $3m$ and γ_{k-1}. In particular, $r \mid ((2+\omega)3m - \gamma_{k-1}) \Leftrightarrow r \mid 1$. Similarly, if r divides both $3m$ and γ_k, then we must have $r \mid (1+\omega)$ noting that $r \mid \big((2+\omega)3m - \gamma_k\big)$. These imply that r is a unit in $\mathbb{Z}[\omega]$ and hence γ_n is relatively prime to both γ_{k-1} and γ_k. With the rest of the labels being successive Eisenstein integers, and $\ell(v_1)$ being relatively prime to all elements of $[\gamma_n]$, then a prime labeling is achieved.

Hence, in any case, W_n admits an Eisenstein prime labeling. ■

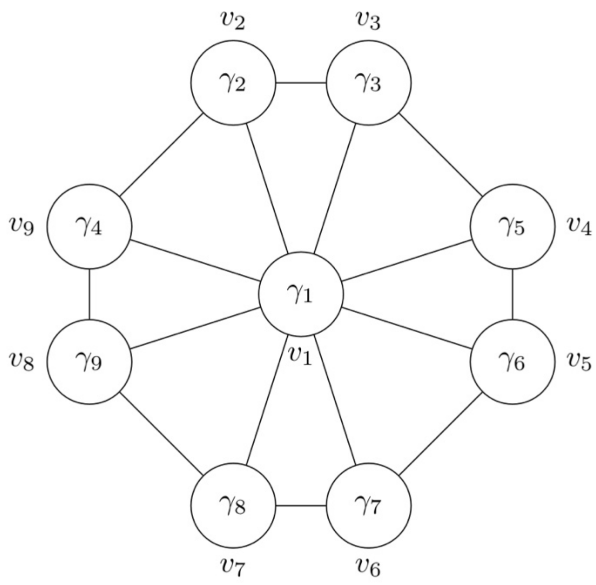

Figure 2: Eisenstein prime labeling for W_9.

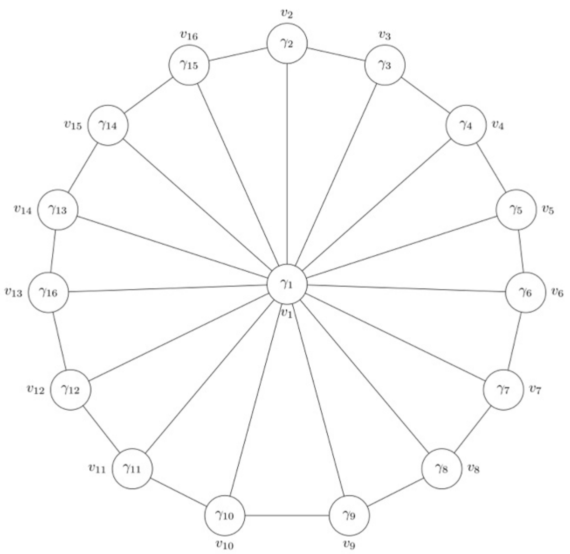

Figure 3: Eisenstein prime labeling for W_{16}.

Definition 4.7. An *n-centipede tree* $c_n = (V, E)$ is a tree with $V = \{v_1, v_2, \ldots, v_{2n}\}$ and $E = \{v_{2i-1}v_{2i} : 1 \le i \le n\} \cup \{v_{2i-1}v_{2i+1} : 1 \le i \le n-1\}$.

As the name suggests, c_n may be viewed as a centipede whose body consists of the path P_n with edges $\{v_{2i-1}v_{2i+1}: 1 \le i \le n-1\}$ and whose legs consists of n copies of P_2 described by the edges $\{v_{2i-1}v_{2i}: 1 \le i \le n\}$. Now, if we label v_i with γ_i, then the even Eisenstein integers would occur on the endvertices of consecutive legs, or on consecutive vertices of the centipede's body. This is due to the nature of even Eisenstein integers occurring along the same diagonal line being two indices away from each other. If the latter situation takes place, one must employ a switching scheme in order to guarantee the prime labeling of c_n as described in the proof of the following theorem.

Theorem 4.4. *Any n-centipede tree c_n admits an Eisenstein prime labeling.*

Proof. Temporarily label v_i with γ_i. Suppose a string of adjacent even Eisenstein integers occur along the vertices $v_{r+2}, v_{r+4}, \ldots, v_{r+s}$ (for some $r, s \in \mathbb{N}$ where r is odd and s is even) which lie on the body of the centipede. In this situation, the labeling is not prime. Whenever this happens, we employ a switching scheme to mend the labeling of the portion of the graph shown in **Error! Reference source not found.**(A). This is done by swapping the labels of v_{r+i} and v_{r+i+1} for $i \in \{2, 4, \ldots, s\}$. The outcome for this switching scheme is illustrated in **Error! Reference source not found.** (B). After applying the switch, the adjacent labels for $v_{r+2}, v_{r+4}, \ldots, v_{r+s}$ are now odd Eisenstein integers two indices away from each other which are relatively prime. Meanwhile, γ_r and γ_{r+3} are now labels for adjacent vertices and so is the pair γ_{r+s+1} and γ_{r+s+2}. The second pair will pose no problem since they are consecutive odd Eisenstein integers. Now, following the definition of our diagonal ordering, $\gamma_{r+2} = k$ where k is the even natural number contained in the kth diagonal. Consequently, $\gamma_r = k - 1 + \omega$ and $\gamma_{r+3} = k + \omega$. Since γ_r and γ_{r+3} differ by a unit, they are relatively prime by Lemma 3.2. and the result follows. ■

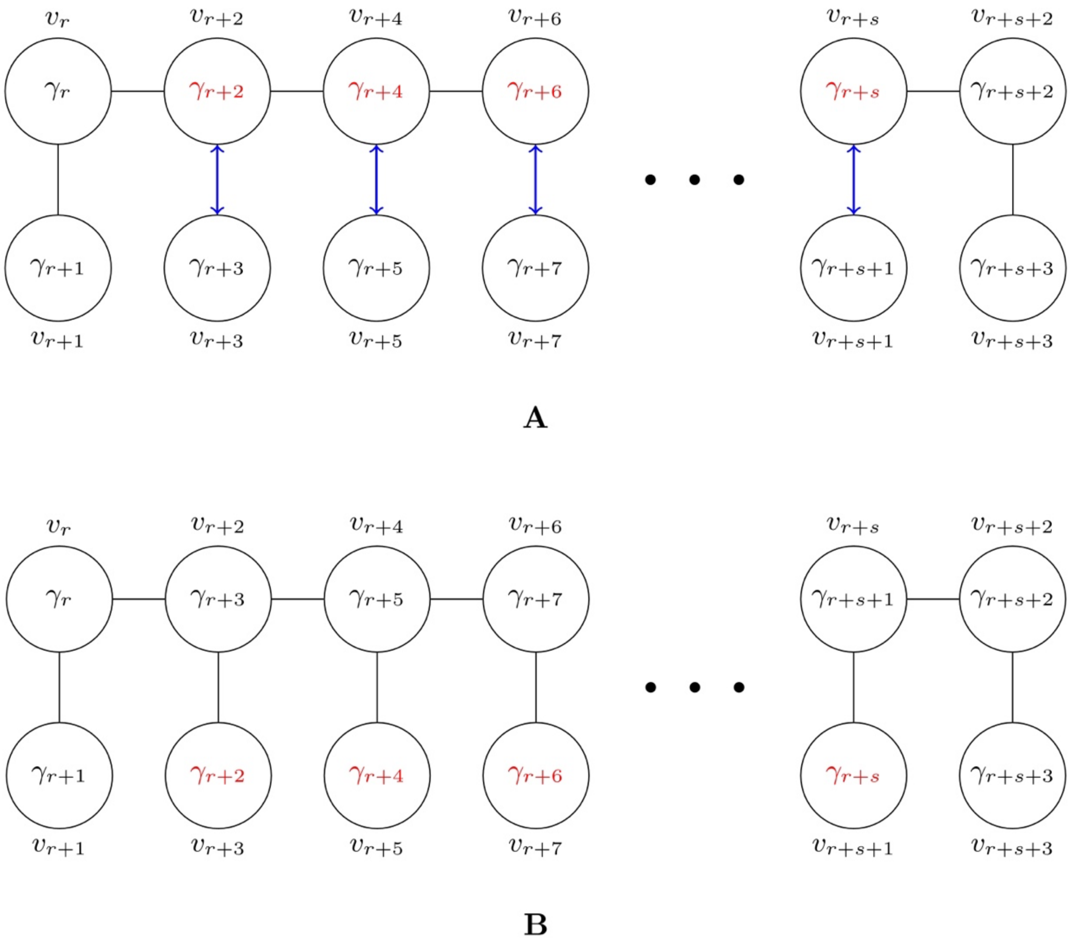

Figure 4: (A) Switching scheme for the string of even Eisenstein integers colored red. (B) Outcome after employing the switching scheme for the string of even Eisenstein integers.

We finish this section by giving prime labelings to double broom graphs.

Definition 4.8. Let k, m, n be positive integers with $k \geq 2$. A *double broom* graph $DB(m,k,n) = (V,E)$ is a graph on $m + k + n$ vertices with $V = \{u_i: 1 \leq i \leq m\} \cup \{v_i: 1 \leq i \leq k\} \cup \{w_i: 1 \leq i \leq n\}$ and $E = \{u_i v_1: 1 \leq i \leq m\} \cup \{v_i v_{i+1}: 1 \leq i \leq k-1\} \cup \{v_k w_i: 1 \leq i \leq n\}$.

Many classes of graphs can be obtained from $DB(m,k,n)$ using specific values of m, k, or n. For instance, $DB(1,k,1)$ is the path graph P_{k+2}. If $n > 1$ (resp. $m > 1$), $DB(1,k,n)$ (resp. $DB(m,k,1)$) is called a *broom graph*. If $k = 2$ and $m, n \geq 3$, the resulting graph $DB(m,2,n)$ is known as the *bistar graph* $B_{m,n}$ which is essentially two star graphs S_m and S_n whose central vertices are joined by an edge.

Theorem 4.5. *If $m + n \geq 6$ and $k \leq \max\{m,n\}$, then the double broom graph $DB(m,k,n)$ admits an Eisenstein prime labeling.*

Proof. From Theorem 3.7. , it can be easily verified that $E(\gamma_{m+k+n}) \leq \frac{m+k+n}{3}$ whenever $m + n \geq 6$ and $k \leq \max\{m,n\}$. This means that $E(\gamma_{m+k+n}) \leq m$ or $E(\gamma_{m+k+n}) \leq n$ or $E(\gamma_{m+k+n}) \leq k$. Let $M = \max\{m,n\}$. In any case, $E(\gamma_{m+k+n}) \leq M$, and $E(\gamma_{m+k+n})$ cannot be greater than both m and n at the same time.

To show that $DB(m,k,n)$ admits an Eisenstein prime labeling, we are going to use the following facts: (i) $\gamma_2 = 2$ is relatively prime to each odd Eisenstein integer; (ii) $\gamma_1 = 1$ is relatively prime to every other Eisenstein integer; and (iii) consecutive Eisenstein integers as well as consecutive odd Eisenstein integers are relatively prime to each other. The main idea is to utilize all the even Eisenstein integers in labeling a portion of the broom whose number of endvertices is greater than $E(\gamma_{m+k+n})$.

Without loss of generality, assume that $M = m$. Define the sets $V_m = \{u_1, \ldots, u_m\}$, $V_k = \{v_2, \ldots, v_{k-1}\}$, and $V_n = \{w_1, \ldots, w_n\}$. Let $\ell(v_1) = 1$ and $\ell(v_k) = 2$. (Note that V_k is empty if $k = 2$.) Use all the even elements in $[\gamma_{m+k+n}] \setminus \{\gamma_2\}$ to label a subset of V_m. The remaining vertices of V_m along with the vertices in $V_k \cup V_n$ shall then be labeled with the odd Eisenstein integers in $[\gamma_{m+k+n}]$ in succession.

To put things more formally, we partition $[\gamma_{m+k+n}]$ using the subsets $E = \{e_j: 1 \leq j \leq E(\gamma_{m+k+n})\}$ and $O = \{f_j: 1 \leq j \leq m + k + n - E(\gamma_{m+k+n})\}$, where e_j (resp. f_j) is the jth even (resp. odd) Eisenstein integer in $[\gamma_{m+k+n}]$ according to the diagonal ordering. The labeling detailed in the previous paragraph is now described by the following rule:

$$\ell(v) = \begin{cases} f_1 = 1, & \text{if } v = v_1 \\ e_1 = 2, & \text{if } v = v_k \\ e_{j+1}, & \text{if } v = u_j \text{ for } 1 \leq j \leq E(\gamma_{m+k+n}) - 1 \\ f_{j-E(\gamma_{m+k+n})+1}, & \text{if } v = u_j \text{ for } E(\gamma_{m+k+n}) + 1 \leq j \leq m \\ f_{m-E(\gamma_{m+k+n})+j}, & \text{if } v = v_j \text{ for } 2 \leq j \leq k-1 \\ f_{m+k-E(\gamma_{m+k+n})+j}, & \text{if } v = w_j \text{ for } 1 \leq j \leq n. \end{cases}$$

We have therefore shown that if $m + n \geq 6$ and $k \leq \max\{m,n\}$, $DB(m,k,n)$ has a prime labeling. ■

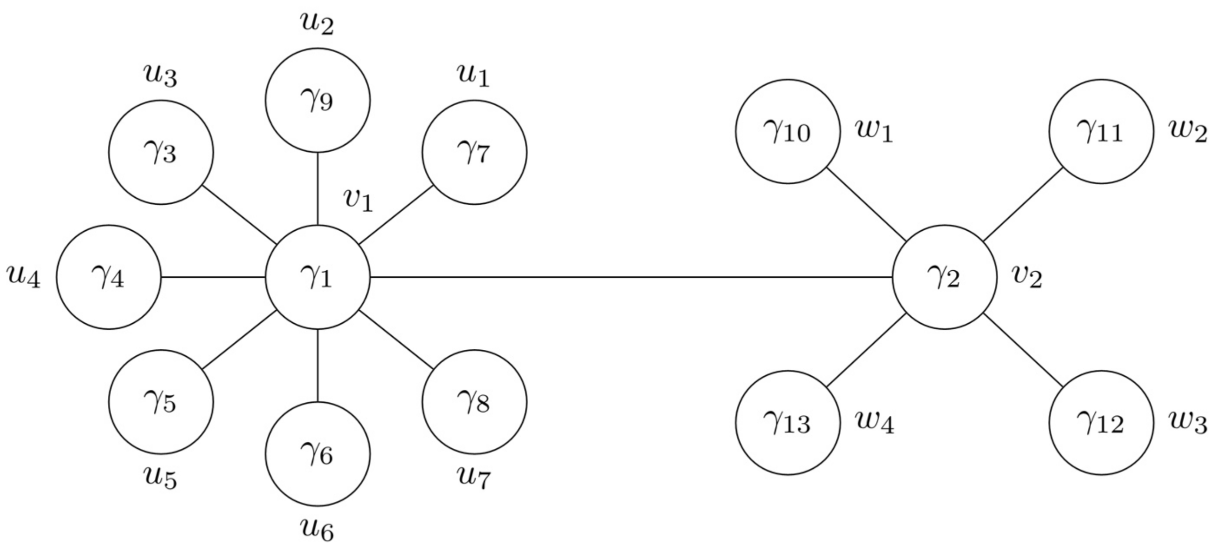

Figure 5: Eisenstein prime labeling for $B_{7,4}$.

As noted earlier, $DB(m,k,n)$ is the bistar graph $B_{m,n}$ whenever $k=2$. Since $m,n \geq 3$ for a bistar graph and $k < 3$, then the conditions of Theorem 4.5. are met. It follows therefore that any bistar graph $B_{m,n}$ also admits a prime labeling. We show in **Error! Reference source not found.** a prime labeling for the bistar graph $B_{7,4}$ while **Error! Reference source not found.** shows a prime labeling for the double broom $DB(7,4,6)$. Both labelings are obtained following the procedure described in the proof of Theorem 4.5.

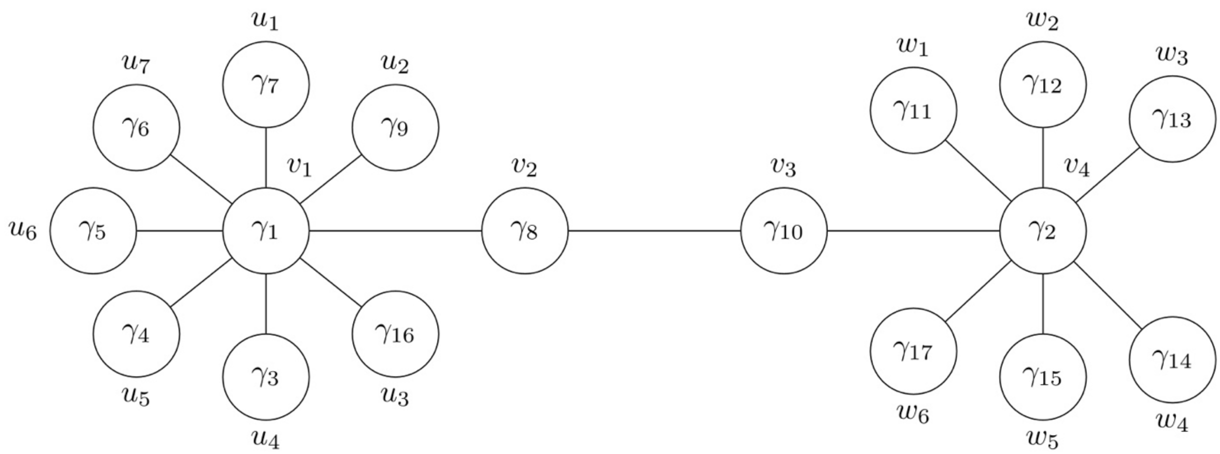

Figure 6: Eisenstein prime labeling for $DB_{7,4,6}$.

Summary and Future Work

In this paper, we defined an order on the set $\mathbb{Z}[\omega]$ of Eisenstein integers that lie in the sector $\left[0, \frac{\pi}{3}\right)$ of the complex plane. Properties of the said ordering were studied and used to arrive at prime labelings of some families of graphs.

Since only a few classes of graphs were considered in this work, one can look at prime labelings of other families of graphs using Eisenstein integers. It is also worth looking into the possibility that Bertrand's Postulate also applies to the set of Eisenstein integers with the defined diagonal ordering. If so, then any tristar graph, and possibly many other types of graphs, can be shown to admit an Eisenstein prime labeling. Another probable area of exploration is on graceful labelings of graphs using Eisenstein integers. Here, one may need to give a modified definition of a graceful graph to fit the nature of the elements of $\mathbb{Z}[\omega]$ and the order on this set.

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